MATH 147 QUIZ 12 SOLUTIONS

1. Use a line integral to find the area contained in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (5pts)

   This problem relies upon Green's theorem. We use the vector field $\mathbf{F}(x, y) = (-\frac{y}{2}, \frac{x}{2})$, as this has Curl $F = 1$. Then, applying Green's theorem, one has

$$\int_C \mathbf{F}\, dr = \iint_S Curl\mathbf{F} \cdot dA = \iint_S d\mathbf{A} = Area(S).$$

   Thus, we calculate this line integral. We parameterize the ellipse in the standard way, that is $r(t) = (a\cos t, b\sin t)$, where t ranges from 0 to 2π. Note that this gives us $r'(t) = (-a\sin t, b\cos t)$. Then, $F(r(t)) = (\frac{-b\sin t}{2}, \frac{a\cos t}{2})$. We now have

$$\int_C \mathbf{F} \cdot ds = \int_0^{2\pi} (\frac{-b\sin t}{2}, \frac{a\cos t}{2}) \cdot (-a\sin t, b\cos t)\, dt = \int_0^{2\pi} ab\sin^2 t/2 + ab\cos^2 t/2\, dt = \int_0^{2\pi} \frac{ab}{2} dt = \pi ab.$$

2. Set $\mathbf{F} = z^2\vec{i} + x^2\vec{j} - y^2\vec{k}$. (5pts)

   (i) Calculate $\nabla \times \mathbf{F}$.

   (ii) Let C be the square path with sides equal to a centered at the point $(x_0, 0, z_0)$ lying in the xz-plane oriented so that each side is parallel to the x or z axis. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

   (iii) Divide your answer in (ii) by the area of the square and take the limit as a goes to zero.

   (iv) . Use your answer in (i) to corroborate your answer in (iii).

   (i)

$$\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ d/dx & d/dy & d/dz \\ z^2 & x^2 & -y^2 \end{vmatrix} = (-2y, 2z, 2x).$$

   (ii) We break down the path C into 4 separate lines for ease of calculation. Let $C_1(t) = (x_0 + \frac{a}{2}, 0, z_0 + at)$, $C_2 = (x_0 - at, 0, z_0 + \frac{a}{2})$, $C_3 = (x_0 - \frac{a}{2}, 0, z_0 - at)$, and $C_4 = (x_0 + at, 0, z_0 - \frac{a}{2})$, all individually with domain $-1/2 \leq t \leq 1/2$. Note this parameterization keeps the surface positively oriented. Note that for all of these lines, $r'(t) = (0, 0, \pm a)$ or $(\pm a, 0, 0)$. Now, we can integrate over the path. One has

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \sum_{i=1}^{4} \int_{C_i} \mathbf{F} \cdot dr \\
&= \int_{-1/2}^{1/2} ((z_0 + at)^2, (x_0 + \frac{a}{2})^2, 0) \cdot (0, 0, a)dt + \int_{-1/2}^{1/2} ((z_0 + a/2)^2, (x_0 - at)^2, 0) \cdot (-a, 0, 0)dt \\
&+ \int_{-1/2}^{1/2} ((z_0 - at)^2, (x_0 - a/2)^2, 0) \cdot (0, 0, -a)dt + \int_{-1/2}^{1/2} ((z_0 - a/2)^2, (x_0 + at)^2, 0) \cdot (a, 0, 0)dt \\
&= \int_{-1/2}^{1/2} -a(z_0 + \frac{a}{2})^2 dt + \int_{-1/2}^{1/2} a(z_0 - \frac{a}{2})^2 dt = -a(z_0 + z_0 a + a^2/4) + a(z_0 2z_0 a + a^2/4) \\
&= -2z_0 a^2
\end{aligned}$$

(iii) Dividing the area gives $-2z_0$, and taking the limit keeps it as $-2z_0$.

(iv) The process of surrounding a point with a path, taking the line integral of said path, dividing by the area, and then using limits to send the area to zero, is precisely the definition of the Curl of a vector field, along a vector normal to the surface enclosed. As the normal vector to our square is $(0, 1, 0)$, it is expected that our answer in part (iii) is the y component of the curl calculated in part (i), once the point $(x_0, 0, z_0)$ is substituted in.